

Multi-point Codes from Generalized Hermitian Curves

Chuangqiang Hu and Chang-An Zhao

Abstract

We investigate multi-point algebraic geometric codes defined from curves related to the generalized Hermitian curve introduced by Alp Bassa, Peter Beelen, Arnaldo Garcia, and Henning Stichtenoth. Our main result is to find a basis of the Riemann-Roch space of a series of divisors, which can be used to construct multi-point codes explicitly. These codes turn out to have nice properties similar to those of Hermitian codes, for example, they are easy to describe, to encode and decode. It is shown that the duals are also such codes and an explicit formula is given. In particular, this formula enables one to calculate the parameters of these codes. Finally, we apply our results to obtain linear codes attaining new records on the parameters. A new record-giving $[234, 141, \geq 59]$ -code over \mathbb{F}_{27} is presented as one of the examples.

Index Terms

algebraic geometric codes, Hermitian codes, order bound.

I. INTRODUCTION

GOPPA constructed error-correcting linear codes by using tools from Algebraic Geometry: a non-singular, projective, geometrically irreducible, algebraic curve \mathcal{X} of genus g defined over \mathbb{F}_q , the finite field with q elements, and two rational divisors D and G on \mathcal{X} . These divisors are chosen in such a way that they have disjoint supports and D equals to a sum of pairwise distinct rational places, $D = P_1 + \dots + P_n$. The algebraic geometric code is defined as

$$C(\mathcal{X}, D, G) := \{(f(P_1), f(P_2), \dots, f(P_n)) : f \in \mathcal{L}(G)\},$$

where $\mathcal{L}(G)$ denotes the Riemann-Roch space associated to G , see [1] as general references for all facts concerning algebraic geometric codes.

One of the main features of Goppa's construction is that the minimum distance is bounded from below, whereas in general there is no lower bound available on the minimum distance of a code. The parameters of an algebraic geometric code are strictly dependent on the curve chosen in the construction. If the curve possesses additional nice properties, one can hope that the corresponding algebraic geometric code also has nice properties. Algebraic geometric codes would advantage one to give an asymptotically good sequence of codes with parameters better than the Varshamov-Gilbert bound in a certain range of the rate and for large enough alphabets.

The most studied codes are probably those arising from the Hermitian curve [1], [2]. The advantage of these codes is that the codes are easy to describe and to encode and decode. Moreover, these codes often have excellent parameters.

One-point codes on Hermitian curves were well-studied in the literature, and efficient methods to decode them were known [1], [3], [4], [5]. The minimum distance of Hermitian two-point codes had been first determined by M. Homma and S. J. Kim [6], [7], [8], [9]. The explicit formulas for the dual minimum distance of such codes were given by S. Park [10]. More recently, Hermitian codes from higher-degree places had been considered in [11]. The dual minimum distance of many three-point codes from Hermitian curves was computed in [12], by extending a recent and powerful approach by A. Couvreur [13]. S. Bulygin investigated one-point codes from the generalized Hermitian curves proposed by Garcia and Stichtenoth; and calculated some parameters of these codes [14]. Some generalizations of these codes were studied by C. Munuera, A. Sepúlveda, and F. Torres [15].

In this paper we investigate multi-point codes from the other generalized Hermitian curves. Let q be a prime power, $\mathbb{F}_{q^{n_0}}$ be the finite field of order q^{n_0} , with $n_0 \geq 2$, and j_0, k_0 be two relatively prime positive numbers such that $j_0 + k_0 = n_0$. We are interested in algebraic geometric codes obtained from the non-singular model \mathcal{X} over $\mathbb{F}_{q^{n_0}}$ of the plane curve

$$\frac{y^{q^{j_0}}}{x} + \frac{y^{q^{j_0+1}}}{x^q} + \dots + \frac{y^{n_0-1}}{x^{q^{k_0-1}}} + \frac{y^1}{x^{q^{k_0}}} + \frac{y^q}{x^{q^{k_0+1}}} + \dots + \frac{y^{q^{j_0-1}}}{x^{q^{n_0}}} = 1, \quad (1)$$

which was introduced by Alp Bassa, Peter Beelen, Arnaldo Garcia, and Henning Stichtenoth [16]. Using Equation (1), they constructed towers of curves of large genera over $\mathbb{F}_{q^{n_0}}$ which are optimal in the sense that they asymptotically attain the

C. Hu and C.-A. Zhao are with the School of Mathematics and Computational Science, School of Mathematics and Computational Science, Sun Yat-sen University, Guangzhou 510275, P.R.China.

E-mail: huchq@mail2.sysu.edu.cn, zhaochan3@mail.sysu.edu.cn

Manuscript received *****; revised *****.

Drinfeld-Vladut bound. For $n_0 = 2, j_0 = k_0 = 1$ this is exactly the Hermitian curve over \mathbb{F}_{q^2} . To see this, we replace xy by z in Equation (1); then

$$z + z^q = x^{q+1}, \quad (2)$$

which is the usual definition of Hermitian curve [1]. Our consideration is the special case with $n_0 = 3, j_0 = 1, k_0 = 2$. According to the paper [16], we introduce four divisors as follows:

- 1) $D := \sum_{\alpha, \beta} D_{\alpha, \beta}$, where $D_{\alpha, \beta} := (x = \alpha, y = \beta)$ with $\alpha, \beta \in \mathbb{F}_{q^3}^*$ satisfies Equation (1);
- 2) $V := (x = 0, y = \infty)$, the divisor consisting of all the places at infinity on the y -axis, which can be written as $V = \sum_{\mu} V_{\mu}$, where $V_{\mu} := (x = 0, y = \infty, x^q y^{q+1} = \mu)$ represents a rational place in V and $\mu^{q-1} = -1$ when q is even;
- 3) $Q := (x = \infty, y = \infty)$, which contains a unique rational place just in case q is odd;
- 4) $P := (x = 0, y = 0)$, the origin of the curve.

The divisors D, V, Q and P contains all the rational places on the curve above. We define the algebraic geometric codes over \mathbb{F}_{q^3} of even characteristic

$$C_r = C(\mathcal{X}, D + P + V, rQ),$$

which are highly similar to the Hermitian codes. Also, we study the multi-point codes over \mathbb{F}_{q^3} of arbitrary characteristic

$$C_{r,s,t} = C(\mathcal{X}, D, rQ + sP + tV).$$

To construct these codes, we find a basis for the Riemann-Roch space $\mathcal{L}(rP + sQ + tV)$ by using Pick's theorem. We use Goppa bound and order bound to estimate the distances of these codes. It turns out that some such codes attain new record values on the parameters. By direct computation, a $[234, 141, \geq 59]$ -code over \mathbb{F}_{27} is presented as one of the examples.

The paper is organized as follows. In Section 2, we introduce some arithmetic properties of the curve (1). The properties of the codes C_r and $C_{r,s,t}$ are presented in Section 3 and Section 4 respectively.

II. THE ARITHMETIC PROPERTIES OF THE CURVE

We follow the notations in Section 1. Let q be a power of a prime and \mathbb{F}_{q^3} be a finite field of cardinality q^3 . In this section we study the curve \mathcal{X} over \mathbb{F}_{q^3}

$$\frac{y^q}{x} + \frac{y^{q^2}}{x^q} + \frac{y}{x^{q^2}} = 1. \quad (3)$$

By the transformation $(x, y) \mapsto (1/y, 1/x)$, we obtain

$$\frac{y^{q^2}}{x} + \frac{y}{x^q} + \frac{y^q}{x^{q^2}} = 1,$$

which is exactly Equation (1) with $j_0 = 2, k_0 = 1$.

Let $P := (x = 0, y = 0)$, $Q := (x = \infty, y = \infty)$, $V := (x = 0, y = \infty)$ be the divisors of the curve \mathcal{X} over \mathbb{F}_{q^3} . Actually, P is a rational place.

Proposition 1. 1) The curve \mathcal{X} has genus $g = (q^4 - 3q + 2)/2$.

2) $\text{div}(x) = \text{div}_0(x) - \text{div}_{\infty}(x) = P + (q+1)V - qQ$, and $\text{div}(y) = \text{div}_0(y) - \text{div}_{\infty}(y) = q^2P - qV - Q$.

3) $\deg(P) = 1, \deg(Q) = q, \deg(V) = q-1$.

4) $v_P(x^{-q^2}y) = 0$, and $x^{-q^2}y \equiv 1 \pmod{P}$.

5) If V_{μ} is a rational place in V , then $v_{V_{\mu}}(x^q y^{q+1}) = 0$, and $x^q y^{q+1} \equiv \mu \pmod{V_{\mu}}$, where $\mu^{q-1} = -1$.

6) If Q_{δ} is a rational place in Q , then $v_{Q_{\delta}}(x^{-1}y^q) = 0$, and $x^{-1}y^q \equiv \delta \pmod{Q_{\delta}}$, where $\delta + \delta^q = 1$.

Proof. The assertions 1), 2), and 3) are shown in [16].

4) The equation $v_P(x^{-q^2}y) = 0$ is clear by using assertion 2). It is easy to show that $v_P(\frac{y^q}{x}) = q^3 - 1 > 0$, and $v_P(\frac{y^{q^2}}{x^q}) = q^4 - q > 0$; then $1 = \frac{y^q}{x} + \frac{y^{q^2}}{x^q} + \frac{y}{x^{q^2}} \equiv \frac{y}{x^{q^2}} \pmod{P}$.

5) Multiplying both sides of Equation (3) by x^{q^2}/y , we get

$$y^{q-1}x^{q^2-1} + (x^q y^{q+1})^{q-1} + 1 = \frac{x^{q^2}}{y}.$$

Again by 2), we have $v_{V_{\mu}}(y^{q-1}x^{q^2-1}) = q^3 - 1 > 0$, and $v_{V_{\mu}}(\frac{x^{q^2}}{y}) = q^3 + q^2 + q > 0$. Hence, we find $(x^q y^{q+1})^{q-1} + 1 \equiv 0 \pmod{V_{\mu}}$.

6) Let $x^{-1}y^q \equiv \delta \pmod{Q_{\delta}}$. Note that $v_{Q_{\delta}}(\frac{y}{x^{q^2}}) = q^3 - 1 > 0$. This implies that $\delta + \delta^q = 1$. \square

We have described the rational places on the curve \mathcal{X} in Section 1. The following corollary gives a simple explanation.

Corollary 2. All the rational places on the curve are the following: $D_{\alpha, \beta}, P, Q_{\delta}$, if q is odd; and $D_{\alpha, \beta}, P, V_{\mu}$, if q is even.

Proof. By Proposition 1, the divisor P is clearly a rational place. Note that the equation $\mu^{q-1} = -1$ has $q-1$ distinct roots in \mathbb{F}_{q^3} for even q while it has no root for odd q . So Proposition 1 implies that the divisor V can be written as $V = \sum_{\mu} V_{\mu}$, where $V_{\mu} := (x = 0, y = \infty, x^q y^{q+1} = \mu)$ represents a rational place in V when q is odd.

Not all the roots of the equation $\delta^q + \delta = 1$ are contained in \mathbb{F}_{q^3} . If $\delta + \delta^q = 1$, then $\delta^q + \delta^{q^2} = 1$, and $\delta^{q^2-1} = 1$. It gives $\delta \in \mathbb{F}_{q^3}^* \cap \mathbb{F}_{q^2}^* = \mathbb{F}_q^*$. Now the equation above becomes $2\delta = 1$. Hence, it has exactly one root in \mathbb{F}_{q^3} if q is odd, and no root if q is even. Therefore, the unique rational place in Q can be written as $Q_{\delta} := (x = \infty, y = \infty, x^{-1}y^q = 1/2)$.

Observe that for all $\alpha \in \mathbb{F}_{q^3}^*$, there exist exactly q^2 distinct elements $\beta \in \mathbb{F}_{q^3}^*$ with

$$\frac{\beta^q}{\alpha} + \frac{\beta^{q^2}}{\alpha^q} + \frac{\beta}{\alpha^{q^2}} = 1. \quad (4)$$

So there is a unique place $D_{\alpha, \beta}$ of degree one such that

$$x \equiv \alpha \pmod{D_{\alpha, \beta}}, \quad y \equiv \beta \pmod{D_{\alpha, \beta}}.$$

Furthermore, the degree of the divisor $D := \sum D_{\alpha, \beta}$ is $\deg D = (q^3 - 1)q^2$. □

Now we come to determine the basis of the Riemann-Roch space $\mathcal{L}(rQ + sP + tV)$. A particularly favorable feature of Hermitian curves is that one can explicitly write a monomial basis for the Riemann-Roch space of a two-point divisor. This is the reason that Hermitian codes are easy to encode and decode. Since the curve \mathcal{X} generalizes the Hermitian curves, we can expect to obtain a monomial basis of $\mathcal{L}(rQ + sP + tV)$. The following proposition is the main result of the paper which can be applied to encoding multi-point codes.

Proposition 3. *The elements $x^i y^j$ with $(i, j) \in \Omega_{r, s, t}$ form a basis of $\mathcal{L}(rQ + sP + tV)$, where*

$$\begin{aligned} \Omega_{r, s, t} := \{ (i, j) \mid & -t \leq (q+1)i - qj < q^3 + q^2 + q - t, \\ & -i - q^2 j \leq s, \\ & qi + j \leq r \}. \end{aligned}$$

We observe that determining the dimension of the Riemann-Roch space is equivalent to calculating the number of lattice points in some region. So our problem becomes a point-counting problem. The proof of the proposition will be given after some preparations.

Lemma 4. *Suppose $A = (x_1, y_1), B = (x_2, y_2)$ are two lattice points on the plane line $l_{r_0} : ax + by = r_0$, where a and b are two integers and coprime; l_A, l_B are two lines pass through A, B respectively and parallel to each other; then for every plane line $l_r : ax + by = r$ parallel to l_{r_0} , the number $\#\sigma_r$ of the lattice points within the segment σ_r between l_A, l_B is a constant $\frac{|x_2 - x_1|}{|b|} = \frac{|y_2 - y_1|}{|a|}$. (The reader shall be careful that we only count once if both two end-points of the segment are lattice points).*

Proof. It is well known that the equation of l_r has integer solutions if and only if r is divisible by the greatest common factor of a and b [17], [18]. Since a and b are coprime, there are two integers x' and y' such that $ax' + by' = 1$. Hence, $ar_0x' + br_0y' = r_0$. So we find a lattice point $(x_0, y_0) := (r_0x', r_0y')$ on the line l_r .

We claim that all the lattice points on the plane line l_r are exactly $\{(x_0 + tb, y_0 - ta) \mid t \in \mathbb{Z}\}$. Clearly, $(x_0 + tb, y_0 - ta)$ satisfies the equation of l_r . Conversely, if $(x, y) \in l_r$ is another lattice point, then

$$a(x - x_0) + b(y - y_0) = 0.$$

Since a and b are coprime, we get $a \mid (y - y_0)$ and $b \mid (x - x_0)$.

Let $C = (x_3, y_3)$ and $D = (x_4, y_4)$ be two end-points of the segment σ_r . Note that the horizontal distance $|x_3 - x_4|$ between C and D is a constant. Moreover, the minimal horizontal distance between two distinct lattice points on σ_r is also a constant b . So the number $\#\sigma_r$ is independent of r . Therefore, we only need to find out all the lattice points on the segment σ_{r_0} . We see that all the lattice points on σ_{r_0} are

$$\begin{aligned} (x_1, y_1) &= A, (x_1 + b, y_1 - a), (x_1 + 2b, y_1 - 2a), \\ \dots, (x_1 + tb, y_1 - ta) &= (x_2, y_2) = B, \end{aligned}$$

where $t = \frac{|x_2 - x_1|}{|b|} = \frac{|y_2 - y_1|}{|a|}$, which implies the lemma. □

Thus, $x^i y^j \in \mathcal{L}(rQ)$ if and only if $0 \leq (q+1)i - qj$, $0 \leq i + q^2 j$, and $qi + j \leq r$. Hence, all the elements in $\{x^i y^j | (i, j) \in \Omega_{rQ}\}$ are contained in $\mathcal{L}(rQ)$.

Similar to the proof of the above lemma, we assume $(i, j) \in \Omega_{rQ}$, then the valuation of $x^i y^j$ at the place P is $i + q^2 j$. The element $x^k y^l$ with the same valuation at P satisfies

$$k = \lambda q^2 + i, \quad l = -\lambda + j.$$

By definition,

$$0 \leq (q+1)i - qj < q^3 + q^2 + q,$$

and

$$(q+1)k - ql = (q+1)i - qj + \lambda(q^3 + q^2 + q).$$

Hence, (k, l) is outside the set Ω_{rQ} for $\lambda \neq 0$. It follows that all the elements in $\{x^i y^j | (i, j) \in \Omega_{rQ}\}$ have different valuations at the place P , therefore they are linearly independent. To complete the proof, we only need to show that the number $\#\Omega_{rQ}$ of the set Ω_{rQ} is exactly the dimension of $\mathcal{L}(rQ)$ for every $r \geq 0$.

Let $A = (q^2 - q, q^2 - 1)$, $B = (q^2, -1)$, and $O = (0, 0)$. Denote by l_A and l_B the parallel lines $(q+1)i - qj = 0$ and $(q+1)i - qj = q^3 + q^2 + q$ respectively; and denote by C and D the intersection points of the line $l_r : qi + j = r$ and the parallel lines l_A, l_B . By definition, the set $\Omega_{r_0 Q}$ contains exactly the lattice points in the trapezoid $OBDC$ except the edge BD . Applying Pick's Theorem [19], [20], the number I of the lattice points in the interior located in the triangle $\triangle OAB$ can be calculated by the formula

$$I = S - M/2 + 1,$$

where S is the area of the triangle $\triangle OAB$, and M is the number of lattice points on the boundary. Let $r_0 := q^3 - 1$. The equation of the line AB is $l_{r_0} : qi + j = r_0$. Note that the lattice points in $\Omega_{r_0 Q}$ are those in the triangle $\triangle OAB$ except the vertex B . Hence, we have

$$\#\Omega_{r_0 Q} = I + M - 1 = S + M/2.$$

It follows from the proof of Lemma 4 that all the lattice points on the segment \overline{OA} are

$$(0, 0) = O, (q, q+1), (2q, 2(q+1)), (3q, 3(q+1)), \\ \dots, ((q-2)q, (q-2)(q+1)), (q^2 - q, q^2 - 1) = A;$$

and those on the segment \overline{AB} are

$$(q^2 - q + 1, (q^2 - 1) - q), (q^2 - q + 2, (q^2 - 1) - 2q), \\ \dots, (q^2 - 1, q - 1), (q^2, -1) = B.$$

Moreover, there is no lattice point on the segment \overline{OB} except the end-points. Hence, $M = 2q$. By direct computation, we find $S = (q^2 - 1/q)q^2/2 = q^4/2 - q/2$. This implies

$$\#\Omega_{r_0 Q} = q^4/2 + q/2.$$

On the other hand, since $\deg(r_0 Q) = q^4 - q > q^4 - 3q = 2g - 2$, by the Riemann-Roch Theorem, we obtain

$$\begin{aligned} \dim \mathcal{L}(r_0 Q) &= 1 - g + \deg(r_0 Q) \\ &= -(q^4 - 3q)/2 + (q^3 - 1)q \\ &= q^4/2 + q/2 = \#\Omega_{r_0 Q}, \end{aligned} \tag{5}$$

and

$$\dim \mathcal{L}((r+1)Q) = \dim \mathcal{L}(rQ) + q, \quad \text{for } r \geq r_0.$$

Note that the set consisting of the lattice points on the segment \overline{CD} is equal to $\Omega_{(r+1)Q} \setminus \Omega_{rQ}$. Now Lemma 4 shows that $\#\Omega_{(r+1)Q} = \#\Omega_{rQ} + q$ for $r \geq r_0$, therefore $\#\Omega_{rQ} = \dim \mathcal{L}(rQ)$.

It remains to consider the case $r < r_0$. Since the lattice point $(i, j) \in \Omega_{rQ} \setminus \Omega_{(r-1)Q}$ represents an element in $\mathcal{L}(rQ) \setminus \mathcal{L}((r-1)Q)$, we obtain

$$\#\Omega_{rQ} - \#\Omega_{(r-1)Q} \leq \dim \mathcal{L}(rQ) - \dim \mathcal{L}((r-1)Q). \tag{6}$$

Sum both sides of Equation (6). Then $\#\Omega_{r_0 Q} \leq \dim \mathcal{L}(r_0 Q)$. Moreover, Equation (5) implies that the equality holds in (6) for $r \leq r_0$. So we conclude that $\#\Omega_{rQ} = \dim \mathcal{L}(rQ)$ for $r \leq r_0$. \square

Corollary 6. *The elements $x^i y^j$ with $(i, j) \in \Omega'_{rQ}$ form a basis of $\mathcal{L}(rQ)$, where*

$$\begin{aligned} \Omega'_{rQ} &:= \{(i, j) | 0 \leq (q+1)i - qj, \quad qi + j \leq r, \\ &\quad -1 < j \leq q^2 - 1\}. \end{aligned}$$

Proof. As the statement in the proof of Proposition 5, all the elements in $\{x^i y^j | (i, j) \in \Omega'_{rQ}\}$ are contained in $\mathcal{L}(rQ)$. According to [16], the polynomial

$$\phi(y) := \frac{y^{q^2}}{x^q} + \frac{y^q}{x} + \frac{y}{x^{q^2}} - 1,$$

over $\mathbb{F}_{q^3}(x)$ is irreducible. Then $1, y^1, y^2, \dots, y^{q^2-1}$ are linearly independent over $\mathbb{F}_{q^3}(x)$. So all the elements contained in $\{x^i y^j | (i, j) \in \Omega'_{rQ}\}$ are linearly independent over \mathbb{F}_{q^3} . Denote the number of the lattice point set Ω'_{rQ} by $\#\Omega'_{rQ}$. It is sufficient to show that $\#\Omega_{rQ} = \#\Omega'_{rQ}$. From the figure below, we see that Ω'_{rQ} (resp. Ω_{rQ}) contains exactly the lattice points in the polygon $OBDC$ (resp. $OBD'C'A$) except the edge BD (resp. BD'); and in particular Ω'_{r_0Q} (resp. Ω_{r_0Q}) contains exactly the lattice points in the triangle $\triangle OAB$ except the vertex B . Thus, $\#\Omega_{r_0Q} = \#\Omega'_{r_0Q}$. Now the corollary follows from Lemma 4. \square

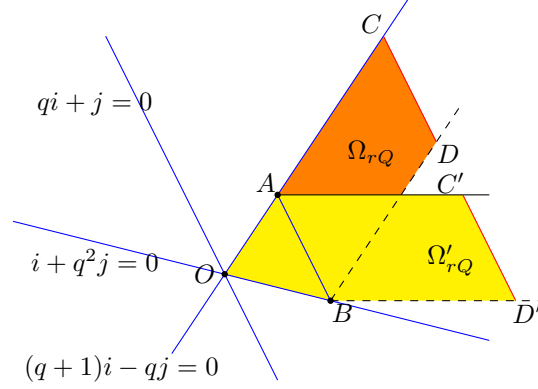


Fig. 3.

Now we are in a position to give the proof of Proposition 3.

Proof of Proposition 3. As shown in the figure below, we have

$$\Omega_{r,s,t} = D'_1 \cup D_2 \cup D_3 \cup D'_4 \cup D'_5 \cup D_6 \cup D_7,$$

and

$$\Omega_{rQ} = D_1 \cup D_2 \cup D_3 \cup D_4.$$

Clearly, $\#D_i = \#D'_i$, so we obtain

$$\begin{aligned} \#\Omega_{r,s,t} &= \#\Omega_{rQ} + \#(D_5 \cup D_6) + \#D_7 \\ &= \#\Omega_{rQ} + s + (q-1)t \\ &= 1 - g + \deg(rQ + sP + tV). \end{aligned}$$

The Riemann-Roch Theorem implies

$$\#\Omega_{r,s,t} = \dim \mathcal{L}(rQ + sP + tV).$$

As the statement in the proof of Proposition 5, we find that $\{x^i y^j | (i, j) \in \Omega_{r,s,t}\}$ is a basis of $\mathcal{L}(rQ + sP + tV)$. \square

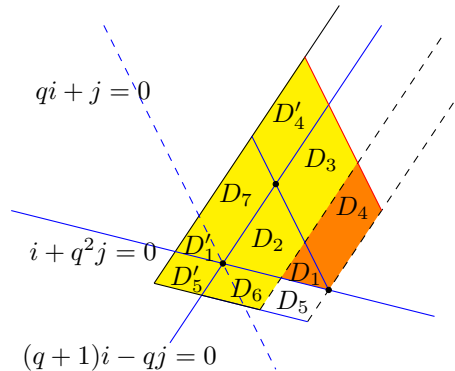


Fig. 4.

We remark that our method in the proof above can be applied to determining the basis of $\mathcal{L}(rQ + sP + tV)$ on the curve (1) with $j_0 = 1, k_0 = n_0 - 1$.

III. THE CODE $C(\mathcal{X}, D + P + V, rQ)$

Let $E := D + P + V$. Throughout this section, we assume that q is even. Then the divisor V consists of rational places by Corollary 2. We study the linear code

$$C_r = C(\mathcal{X}, E, rQ).$$

The length of C_r is $n_1 := \deg(E) = \deg(D + P + V) = 1 + q - 1 + (q^3 - 1)q^2 = q(q^4 - q + 1)$. It is well known that the dimension of an algebraic geometric code $C(\mathcal{X}, E, G)$ is given by

$$\dim C_r = \dim \mathcal{L}(G) - \dim \mathcal{L}(G - E). \quad (7)$$

Let $R_1 := (n_1 + 2g - 2)/q = q^4 + q^3 - q - 2$. For $r > R_1$, the Riemann-Roch Theorem and Equation (7) yield

$$\begin{aligned} \dim C_r &= (1 - g + \dim(rQ)) - (1 - g + \deg(rQ - E)) \\ &= \deg E = n_1. \end{aligned}$$

Hence $C_r = \mathbb{F}_{q^3}^{n_1}$ in this case which is trivial. So we should only consider the case $0 \leq r \leq R_1$.

Definition 7. Two codes $C_1, C_2 \subseteq \mathbb{F}_{q^3}^{n_1}$ are said to be **equivalent** if there is a vector $a = (a_1, a_2, \dots, a_n) \in (\mathbb{F}_{q^3}^*)^n$ such that $C_2 = a \cdot C_1$; i.e.,

$$C_2 = \{(a_1 c_1, a_2 c_2, \dots, a_n c_n) \mid (c_1, c_2, \dots, c_n) \in C_1\}.$$

Denote by C^\perp the dual of C . The code C is called **self-dual** (resp. **self-orthogonal**) if $C = C^\perp$ (resp. $C \subseteq C^\perp$). The code C is called **self-equivalent** if C is equivalent to C^\perp .

We need the following lemma which is shown in [1].

Lemma 8 ([1]). *Let η be a Weil differential such that $v_{P_i}(\eta) = -1$ for $i = 1, \dots, n$. Then the dual of $C(\mathcal{X}, D, G)$ is equivalent to the code $C(\mathcal{X}, D, D - G + \text{div}(\eta))$. Furthermore, denote by $\text{res}_P(\eta)$ the residue of η at P , then each vector in $C(\mathcal{X}, D, G)^\perp$ can be written as*

$$(\text{res}_{P_1}(\eta)c_1, \text{res}_{P_2}(\eta)c_2, \dots, \text{res}_{P_n}(\eta)c_n),$$

where $(c_1, c_2, \dots, c_n) \in C(\mathcal{X}, D, G)$. Moreover, if $\text{res}_{P_i}(\eta) = 1$, then the dual of $C(\mathcal{X}, D, G)$ is equal to $C(\mathcal{X}, D, D - G + \text{div}(\eta))$.

Proposition 9. *The dual of C_r is*

$$C_r^\perp = C_{R_1 - r}.$$

Hence C_r is self-orthogonal if $2r \leq R_1$, and C_r is self-dual for $r = R_1/2$.

Proof. Proposition 1 shows

$$\begin{aligned} \text{div}(x) &= \text{div}_0(x) - \text{div}_\infty(x) \\ &= P + (q + 1)V - qQ, \end{aligned}$$

and

$$\begin{aligned} \text{div}(y) &= \text{div}_0(y) - \text{div}_\infty(y) \\ &= q^2 P - qV - Q. \end{aligned}$$

Consider the element

$$t := \prod_{\alpha \in \mathbb{F}_{q^3}} (x - \alpha) = x^{q^3} - x.$$

Then t is a prime element for all places $D_{\alpha, \beta}$, and its divisor is

$$\begin{aligned} \text{div}(t) &= \text{div}_0(x) + D - q^3 \text{div}_\infty(x) \\ &= P + (q + 1)V + D - q^4 Q. \end{aligned}$$

The differential dt has the divisor

$$\begin{aligned} \text{div}(dt) &= \text{div}(-dx) = -2 \text{div}_\infty(x) + \text{Diff}(F/K(x)) \\ &= -2qQ + (q^3 + q - 2)Q + qV \\ &= (q^3 - q - 2)Q + qV, \end{aligned}$$

where we use the formula $\text{Diff}(F/K(x)) = (q^3 + q - 2)Q + qV$ according to [16]. Let $\eta := dt/t$ be a Weil differential. The divisor of η is

$$\begin{aligned} \text{div}(\eta) &= \text{div}(dt) - \text{div}(t) \\ &= (q^3 - q - 2)Q + qV \\ &\quad - P - (q + 1)V - D + q^4Q \\ &= -P - V - D + (q^4 + q^3 - q - 2)Q \\ &= -P - V - D + R_1Q. \end{aligned}$$

Clearly, the Weil differential η satisfies the condition in Lemma 8; therefore the dual C_r^\perp of C_r is equivalent to

$$\begin{aligned} C(\mathcal{X}, E, E - rQ + \text{div}(\eta)) &= C(\mathcal{X}, E, P + D + V - rQ \\ &\quad - P - V - D + R_1Q) \\ &= C(\mathcal{X}, E, (R_1 - r)Q) \\ &= C_{R_1 - r}. \end{aligned}$$

It remains to show that $\text{res}_{P'}(\eta) = 1$, for cases $P' = P$, $P' = D_{\alpha, \beta}$, and $P' = V_\mu$. Only the last case $P' = V_\mu$ is non-trivial. Define $z := xy$. Then z is prime at V_μ and its divisor is

$$\begin{aligned} \text{div}(z) &= \text{div}(x) + \text{div}(y) \\ &= P + (q + 1)V - qQ + q^2P - qV - Q \\ &= (1 + q^2)P + V - (q + 1)Q. \end{aligned}$$

By Equation (3),

$$x^{q+q^2}y^q + x^{1+q^2}y^{q^2} + x^{1+q}y = x^{1+q+q^2}.$$

Replacing xy by z , we get

$$x^{q^2}z^q + xz^{q^2} + x^qz = x^{1+q+q^2}. \quad (8)$$

The differential of Equation (8) is

$$-\left(z^{q^2} - (1 + q + q^2)x^{q+q^2}\right)dx = x^qdz.$$

This implies

$$\begin{aligned} \eta &= \frac{dt}{t} = \frac{-dx}{x^{q^3} - x} \\ &= \frac{x^q dz}{(z^{q^2} - (1 + q + q^2)x^{q+q^2})(x^{q^3} - x)} \\ &= \frac{x^{q-1} dz}{(z^{q^2} + *)(-1 + *)} \\ &= \left(-\frac{x^{q-1}}{z^{q^2-1}} + *\right) \frac{dz}{z}. \end{aligned}$$

By Proposition 1, we obtain $\frac{x^{q-1}}{z^{q^2-1}} = (x^q y^{q+1})^{q-1} \equiv -1 \pmod{V_\mu}$. Hence, $\text{res}_{V_\mu}(\eta) = \text{res}_{V_\mu}(dt/t) = 1$. Now, Lemma 8 shows $C_r^\perp = C_{R_1 - r}$. \square

Proposition 10. Suppose that $0 \leq r \leq R_1$. Then the following holds:

1) The dimension of C_r is given by

$$\dim C_r = \begin{cases} \#\Omega'_{rQ} & \text{for } 0 \leq r < q^4 - q + 1, \\ n_1 - \#\Omega'_{sQ} = q(q^4 - q + 1) - \#\Omega'_{sQ} & \text{for } q^4 - q + 1 \leq r \leq R_1. \end{cases}$$

where $s = R_1 - r$.

2) For $q^3 - 3 < r < q^4 - q + 1$ we have $\dim C_r = qr - (q^4 - 3q)/2$.

3) The minimum distance d of C_r satisfies $d \geq q(q^4 - q + 1 - r)$.

Proof. 1) For $0 \leq r < n_1/q = q^4 - q + 1$, the inequality $\deg(rQ - E) < 0$ implies $\mathcal{L}(E - rQ) = 0$. By Proposition 6 and Equation (7), we get

$$\dim C_r = \dim \mathcal{L}(rQ) = \#\Omega'_{rQ}.$$

For $q^4 - q + 1 \leq r \leq q^4 + q^3 - q - 2$, we set $s := q^4 + q^3 - q - 2 - r$, then

$$0 \leq s = q^4 + q^3 - q - 2 - r \leq q^3 - 3 < q^4 - q + 1.$$

Proposition 9 yields

$$\dim C_r = n_1 - \dim C_s = q(q^4 - q + 1) - \#\Omega'_{rQ}.$$

2) For $q^3 - 3 < r < q^4 - q + 1$, $\deg(rQ) > 2g - 2$, so the Riemann-Roch Theorem gives

$$\dim C_r = qr + 1 - g = qr - (q^4 - 3q)/2.$$

3) The inequality $d \geq q(q^4 - q + 1 - r)$ follows from Goppa bound. □

Proposition 11. Suppose $0 \leq r < q^4 - q + 1$.

- 1) If $r = qt$, $t \leq q^3 - 1$, then $d = q(q^4 - q + 1 - r)$.
- 2) If $r = 1 + q$, then $d = q(q^4 - q + 1 - r)$.
- 3) If $r = 1 + qt$, $0 < t \leq q^3 - q^2$, then $d = q(q^4 - q + 1 - r)$.
- 4) If $r = (1 + q) + (q^3 + q^2 + q)t$, $0 \leq t \leq q - 2$, then $d = q(q^4 - q + 1 - r)$.

Proof. 1) Choose t distinct elements $\alpha_1, \dots, \alpha_t \in \mathbb{F}_{q^3}^*$, and consider the element

$$z := \prod_{i=1}^t (x - \alpha_i).$$

Its divisor is

$$\operatorname{div}(z) = -qtQ + \sum_{i=1}^t D_{\alpha_i},$$

where D_{α_i} denotes the divisor $D_{\alpha_i} := \sum_{\beta} D_{\alpha_i, \beta}$. Therefore, $z \in \mathcal{L}(rQ)$. Note that the element z has exactly $q^2 t$ distinct zeros $D_{\alpha_i, \beta}$ of degree one, so the weight of the corresponding codeword $ev_D(z) \in C_r$ is $q(q^4 - q + 1 - qt)$.

2) Fix an element $\beta \in \mathbb{F}_{q^3}^*$, and consider

$$z := x(y - \beta).$$

By the strict triangle inequality, we have

$$\operatorname{div}(y - \beta) = -Q + D_{\beta} - qV,$$

where D_{β} denotes the divisor $D_{\beta} := \sum_{\alpha} D_{\alpha, \beta}$ with α, β satisfying Equation (4). The degree of D_{β} is $\deg(D_{\beta}) = q^2$. We find the divisor of z is

$$\operatorname{div}(z) = -(q+1)Q + D_{\beta} + P + V.$$

3) Let $\beta \in \mathbb{F}_{q^3}^*$, and A be the set $\{\alpha | (\alpha, \beta) \notin D_{\beta} \text{ be the solution of Equation (4)}\}$. Since the cardinality of A is $\#A = q^3 - 1 - q^2$, we can choose $s = t - 1$ distinct elements $\alpha_i \in A$. Consider

$$z := x(y - \beta) \prod_{i=1}^s (x - \alpha_i).$$

The divisor of z is

$$\operatorname{div}(z) = -(qt+1)Q + D_{\beta} + P + V + \sum_{i=1}^s D_{\alpha_i}.$$

4) If $0 \leq t \leq q - 2$, then $1 + (q^2 + q + 1)t \leq q^3 - 1$. We can choose $s = 1 + (q^2 + q + 1)t$ distinct elements $\beta_i \in \mathbb{F}_{q^3}^*$ and consider

$$z := x^{1+q^2t} y^{-t} \prod_{i=1}^s (y - \beta_i).$$

Its divisor is

$$\operatorname{div}(z) = -(q+1 + (q^3 + q^2 + q)t)Q + \sum_{i=1}^s D_{\beta_i} + P + V.$$

□

Proposition 12. Suppose that $q^4 + q^3 - q^2 - 2q - 3 < r \leq R_1$, then the minimum distance of C_r is $d = 2$.

Proof. The element in Ω_{rQ} with least valuation at Q , and valuation 0 at V is $x^q y^{q+1}$. Let $r_1 := -v_Q(x^q y^{q+1}) = q^2 + q + 1$. Assume that $q^4 + q^3 - q^2 - 2q - 3 < r \leq R_1$. By Proposition 9, the dual of C_r is C_{R_1-r} with $0 \leq R_1 - r < q^2 + q + 1 = r_1$. According to Corollary 6, the basis of C_{R_1-r} can be given by $f_1 = 1, f_2 = x, f_3 = xy, \dots, f_k$, where k denotes the dimension of C_{R_1-r} . Applying Proposition 1, we see that $f_i(P) = 0$ and $f_i(V) = 0$ for $2 \leq i \leq k$. Therefore the check matrix of C_r is given by

$$H = \begin{bmatrix} & P & V \\ * & * & 1 & 1 \\ * & * & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ * & * & 0 & 0 \end{bmatrix}.$$

Let $c := (0, 0, \dots, 0, 1, -1)$. Then $Hc^T = \mathbf{0}$. Hence, $c \in C_r$, and $d(C_r) \leq w(c) = 2$. Note that $d(C_r) \geq 2$, then $d(C_r) = 2$. \square

Example 13. Let us consider the case $q = 2$, then $n_1 = 30$, $g = 6$, and $R_1 = 20$. By Proposition 11 and Corollary (6), we obtain the following table.

TABLE I

r	dim	basis	d	r	dim	basis	d
0	1	1	30	11	17	$x^5 y, x^4 y^3$	8
2	2	x	26	12	19	$x^6, x^5 y^2$	6
3	3	xy	24	13	21	$x^6 y, x^5 y^3$	5
4	4	x^2	22	14	23	$x^7, x^6 y^2$	2
5	5	$x^2 y$	20	15	25	$x^7 y, x^6 y^3$	2
6	7	$x^3, x^2 y^2$	18	16	26	$x^7 y^2$	2
7	9	$x^3 y, x^2 y^3$	16	17	27	$x^7 y^3$	2
8	11	$x^4, x^3 y^2$	14	18	28	$x^8 y^2$	2
9	13	$x^4 y, x^3 y^3$	12	19	29	$x^8 y^3$	2
10	15	$x^5, x^4 y^2$	10	21	30	$x^9 y^3$	1

(Note: $C_1 = C_0$, $C_{20} = C_{19}$)

The distance with $r = 11$ or 13 is not easy to calculate. By direct computation, $d(C_{13}^{(1)}) = 5$. The function

$$\begin{aligned} \varphi(x) := & 1 + x + xy + x^2 + x^2 y + x^2 y^2 + x^3 + 0 + x^3 y + 0 \\ & + x^4 + x^3 y^3 + 0 + x^4 y^2 + x^5 + x^4 y^3 + x^5 y, \end{aligned}$$

achieve the Goppa bound with $r = 11$. Then, $d(C_{11}^{(1)}) = 8$.

IV. THE CODE $C(\mathcal{X}, D, rQ + sP + tV)$

In this section we study the code

$$C_{r,s,t} := C_{\mathcal{L}}(D, rQ + sP + tV).$$

The length of $C_{r,s,t}$ is $n_2 := \deg(D) = (q^3 - 1)q^2$.

Proposition 14 ([1]). *Suppose G_1 and G_2 are divisors with $G_1 \sim G_2$ and $\text{supp } G_1 \cap \text{supp } D = \text{supp } G_2 \cap \text{supp } D = \emptyset$, then $C(\mathcal{X}, D, G_1)$ and $C(\mathcal{X}, D, G_2)$ are equivalent.*

Let $C_{r,s} := C_{r,s,0}$ be the code $C_{r,s,t}$ with $t = 0$. We observe that the code $C_{r,s,t}$ is equivalent to $C_{r',s'}$ for some $r', s' \in \mathbb{Z}$.

Proposition 15. 1) *The code $C_{r,s,t}$ is equivalent to $C_{r+t(q+1), s-t(q^2+1)}$.*

2) *The code $C_{r,s}$ is equivalent to $C_{r-(q^2+q+1), s+(q^2+q+1)q}$.*

Therefore, $C_{r,s,t}$ can be written as $C_{r',s'}$ up to equivalence, where $0 \leq r' < (q^2 + q + 1)$, $s' \geq 0$.

Proof. 1) Applying Proposition 1, we obtain

$$\text{div}(xy) = V + (q^2 + 1)P - (q + 1)Q,$$

which means that the divisor V is equivalent to $(q + 1)Q - (q^2 + 1)P$. Hence,

$$rQ + sP + tV \sim (r + t(q + 1))Q + (s - t(q^2 + 1))P.$$

Proposition 14 yields that the code $C_{r,s,t}$ is equivalent to $C_{r+t(q+1), s-t(q^2+1)}$.

2) By Proposition 1 again, we get

$$\operatorname{div}(x^q y^{q+1}) = -(q^2 + q + 1)Q + (q^2 + q + 1)qP.$$

So we have

$$(q^2 + q + 1)Q \sim (q^2 + q + 1)qP.$$

Then,

$$rQ + sP \sim (r - (q^2 + q + 1))Q + (s + (q^2 + q + 1)q)P.$$

Hence, the code $C_{r,s}$ is equivalent to $C_{r-(q^2+q+1), s+(q^2+q+1)q}$ by Proposition 14. \square

Proposition 16. *Up to equivalence, the dual space of $C_{r,s}$ is*

$$\begin{cases} C_{q^2-1-r, q^5+q^4-q^3-q^2-2q-s} & \text{for } 0 \leq r \leq q^2-1, \\ C_{2q^2+q-r, q^5+q^4-2q^3-2q^2-3q-s} & \text{for } q^2 \leq r \leq q^2+q. \end{cases}$$

Hence, the code $C_{q^2+q/2, q^5/2+q^4/2-q^3-q^2-3q/2}$ is self-equivalent for even q ; and the code $C_{(q^2-1)/2, (q^5+q^4-q^3-q^2-2q)/2}$ is self-equivalent for odd q .

Proof. We follow the notations in the proof of Proposition 9,

$$\operatorname{div}(\eta) = (q^4 + q^3 - q - 2)Q - V - P - D.$$

Let $\bar{\eta} := xy(x^q y^{q+1})^{q^2-2}\eta$ be a Weil differential on \mathcal{X} . Recall that

$$\begin{aligned} \operatorname{div}\left((x^q y^{q+1})^{q^2-2}\right) &= (q^2 - 2)(q^3 + q^2 + q)P \\ &\quad - (q^2 - 2)(q^2 + q + 1)Q, \end{aligned}$$

and

$$\operatorname{div}(xy) = V + (q^2 + 1)P - (q + 1)Q.$$

Thus, the divisor of $\bar{\eta}$ is given by

$$\begin{aligned} \operatorname{div}(\bar{\eta}) &= (q^4 + q^3 - q - 2)Q - V - P - D \\ &\quad + V + (q^2 + 1)P - (q + 1)Q \\ &\quad + (q^2 - 2)((q^3 + q^2 + q)P - (q^2 + q + 1)Q) \\ &= -D + (q^2 - 1)Q + (q^5 + q^4 - q^3 - q^2 - 2q)P. \end{aligned}$$

Proposition 15 yields

$$\begin{aligned} D - sP - rQ + \operatorname{div}(\bar{\eta}) &= (q^2 - 1 - r)Q + (q^5 + q^4 - q^3 - q^2 - 2q - s)P \\ &\sim (2q^2 + q - r)Q + (q^5 + q^4 - 2q^3 - 2q^2 - 3q - s)P. \end{aligned}$$

By Lemma 8, we obtain

$$\begin{aligned} C_{r,s}^\perp &\cong C_{q^2-1-r, q^5+q^4-q^3-q^2-2q-s} \\ &\cong C_{2q^2+q-r, q^5+q^4-2q^3-2q^2-3q-s}. \end{aligned}$$

\square

As in Section 2, we set $R_2 := n_2 + 2g - 2 = q^5 + q^4 - q^2 - 3q$. We will be interested in the case, when $0 \leq rq + s \leq R_2$. Set $\Omega_{r,s} := \Omega_{r,s,0}$, where $\Omega_{r,s,t}$ is defined in Proposition 3. Next we investigate the dimension and the distance of the code $C_{r,s}$.

Proposition 17. *Suppose that $0 \leq rq + s \leq R_2$. Then the following holds:*

1) *The dimension of $C_{r,s}$ is given by*

$$\dim C_{r,s} = \begin{cases} \#\Omega_{r,s} & \text{for } 0 \leq rq + s < n_2, \\ n_2 - \#\Omega_{r,s}^\perp = (q^5 - q^2) - \#\Omega_{r,s}^\perp & \text{for } n_2 \leq rq + s \leq R_2, \end{cases}$$

where $\Omega_{r,s}^\perp$ is defined by

$$\Omega_{r,s}^\perp := \begin{cases} \Omega_{q^2-1-r, q^5+q^4-q^3-q^2-2q-s} & \text{for } 0 \leq r \leq q^2-1, \\ \Omega_{2q^2+q-r, q^5+q^4-2q^3-2q^2-3q-s} & \text{for } q^2 \leq r \leq q^2+q. \end{cases}$$

2) For $q^4 - 3q < rq + s < q^5 - q^2$ we have $\dim C_r = qr + s - (q^4 - 3q)/2$.

3) The minimum distance d of C_r satisfies $d \geq n_2 - rq - s = q^5 - q^2 - rq - s$.

Proof. 1) For $0 \leq rq + s < n_2$, Proposition 3 and Equation (7) give

$$\dim C_{r,s} = \dim \mathcal{L}(rQ + sP) = \#\Omega_{r,s}.$$

By Proposition 16, the dual of the code $C_{r,s}$ is equivalent to $C_{r',s'}$ for some $r', s' \in \mathbb{Z}$. For $n_2 \leq rq + s \leq R_2$, we obtain

$$\begin{aligned} \dim C_{r,s} &= n_2 - \dim C_{r',s'} \\ &= (q^5 - q^2) - \#\Omega_{r',s'}^\perp. \end{aligned}$$

2) Assume that $q^4 - 3q = 2g - 2 < rq + s < n_2 = q^5 - q^2$. Then, $\deg(rQ + sP) > 2g - 2$, so the Riemann-Roch Theorem gives

$$\begin{aligned} \dim C_{r,s} &= \deg(rQ + sP) + 1 - g \\ &= qr + s + 1 - g \\ &= qr + s - (q^4 - 3q)/2. \end{aligned}$$

3) The inequality follows immediately from Goppa bound. □

Proposition 18. Suppose $0 \leq qr + s < n_2$.

- 1) If $s = (q^3 + q^2 + q)(q\tau - r)$, $0 \leq (q^2 + q + 1)\tau - qr - r \leq q^3 - 1$, $0 \leq q\tau - r$, then $d = n_2 - rq - s$.
- 2) If $r = 0$, $s = q^2$, then $d = n_2 - rq - s$.
- 3) If $s = (q\tau - r)(q^3 + q^2 + q) + \lambda q^2$, $0 \leq (q^2 + q + 1)\tau - qr - r \leq q^3 - 1 - \lambda q^2$, $0 \leq \lambda$, then $d = n_2 - rq - s$.
- 4) If $r = 0$, $s = q^2(q^3 - 1 - \lambda)$, $\lambda q^2 \leq q^3 - 1$, then $d = n_2 - rq - s$.
- 5) If $s = (q^3 + q^2 + q)((q - 1)q - (q\tau + r))$, $0 \leq (q^2 + q + 1)\tau + qr + r \leq q^3 - 1$, then $d = n_2 - rq - s$.

Proof. 1) Choose $\kappa := (q^2 + q + 1)\tau - qr - r$ distinct elements $\alpha_1, \dots, \alpha_\kappa \in \mathbb{F}_{q^3}^*$, and consider

$$z := (x^q y^{q+1})^{r-q\tau} \prod_{i=1}^{\kappa} (x - \alpha_i).$$

The divisor of z is

$$\begin{aligned} \operatorname{div}(z) &= (r - q\tau)((q^3 + q^2 + q)P \\ &\quad - (q^2 + q + 1)Q) \\ &\quad - q\kappa Q + \sum_{i=1}^{\kappa} D_{\alpha_i} \\ &= (-q\kappa - (r - q\tau)(q^2 + q + 1))Q \\ &\quad - (q\tau - r)(q^3 + q^2 + q)P + \sum_{i=1}^{\kappa} D_{\alpha_i} \\ &= -rQ - (q\tau - r)(q^3 + q^2 + q)P + \sum_{i=1}^{\kappa} D_{\alpha_i}. \end{aligned}$$

2) Fix an element $\beta \in \mathbb{F}_{q^3}^*$. We consider

$$z := y^{-1}(y - \beta).$$

Recall that the divisor of $y - \beta$ is

$$\operatorname{div}(y - \beta) = -Q + D_\beta - qV.$$

Then

$$\begin{aligned}\operatorname{div}(z) &= -Q + D_\beta - qV + qV - q^2P + Q \\ &= D_\beta - q^2P.\end{aligned}$$

- 3) Let $\kappa := (q^2 + q + 1)\tau - qr - r$ with $0 \leq \kappa \leq q^3 - 1 - \lambda q^2$. Choose λ distinct elements $\beta_1, \dots, \beta_\lambda$ in $\mathbb{F}_{q^3}^*$, and κ distinct elements $\alpha_1, \dots, \alpha_\kappa$ in the set $A := \{\alpha \in \mathbb{F}_{q^3}^* \mid (\alpha, \beta) \notin D_{\beta_i}, i = 1, \dots, \lambda\}$. Consider

$$z := y^{-\lambda} \prod_{i=1}^{\lambda} (y - \beta_i) (x^{q+1} y^q)^{r-q\tau} \prod_{i=1}^{\kappa} (x - \alpha_i).$$

Then the divisor of z is

$$\begin{aligned}\operatorname{div}(z) &= -\lambda q^2 P + \sum_{i=1}^{\lambda} D_{\beta_i} - q\kappa Q + \sum_{i=1}^{\kappa} D_{\alpha_i} \\ &\quad + (r - q\tau) ((q^3 + q^2 + q)P - (q^2 + q + 1)Q) \\ &= -rQ - ((q\tau - r)(q^3 + q^2 + q) + \lambda q^2) P \\ &\quad + \sum_{i=1}^{\kappa} D_{\alpha_i} + \sum_{i=1}^{\lambda} D_{\beta_i}.\end{aligned}$$

- 4) Suppose that $\mathbb{F}_{q^3}^* = \{\alpha_1, \dots, \alpha_{q^3-1}\}$, and $\beta_1, \dots, \beta_\lambda$ are distinct elements in $\mathbb{F}_{q^3}^*$. Consider

$$z := y^\lambda \prod_{i=1}^{\lambda} (y - \beta_i)^{-1} (x^q y^{q+1})^{q-q^2} \prod_{i=1}^{q^3-1} (x - \alpha_i),$$

where $\lambda q^2 \leq q^3 - 1$. Then its divisor is

$$\begin{aligned}\operatorname{div}(z) &= -\sum_{i=1}^{\lambda} D_{\beta_i} + \lambda q^2 P + q(q^3 - 1)Q \\ &\quad - q^2(q^3 - 1)P - q(q^3 - 1)Q + D \\ &= D - \sum_{i=1}^{\lambda} D_{\beta_i} - q^2(q^3 - 1 - \lambda)P.\end{aligned}$$

- 5) Let $\epsilon = (q^2 + q + 1)\tau + qr + r$. Suppose that $\mathbb{F}_{q^3}^* = \{\beta_1, \dots, \beta_{q^3-1}\}$, and $\alpha_1, \dots, \alpha_\epsilon$ are distinct elements in $\mathbb{F}_{q^3}^*$. Consider

$$z := y^{q^3-1} \prod_{i=1}^{q^3-1} (y - \beta_i) (x^q y^{q+1})^{q\tau+r} \prod_{i=1}^{\epsilon} (x - \alpha_i)^{-1}.$$

Its divisor is

$$\begin{aligned}\operatorname{div}(z) &= \sum_{i=1}^{q^3-1} D_{\beta_i} - (q^3 - 1)q^2 P \\ &\quad - (q^2 + q + 1)(q\tau + r)Q \\ &\quad + (q^2 + q + 1)q(q\tau + r)P + q\epsilon Q - \sum_{i=1}^{\epsilon} D_{\alpha_i} \\ &= D - \sum_{i=1}^{\epsilon} D_{\alpha_i} - rQ \\ &\quad - (q^2 + q + 1)q((q - 1)q - (q\tau + r))P.\end{aligned}$$

□

For a fixed r , we define the Weierstrass set

$$H_r = \{s \in \mathbb{Z} \mid \mathcal{L}(rQ + sP) \neq \mathcal{L}(rQ + (s - 1)P)\}.$$

Proposition 5 yields

$$H_r = \{s \in \mathbb{Z} \mid \Omega_{r,s} \neq \Omega_{r,s-1}\}.$$

Assume that $f \in \mathcal{L}(rQ + sP) \setminus \mathcal{L}(rQ + (s-1)P)$, and $g \in \mathcal{L}(s'P) \setminus \mathcal{L}((s'-1)P)$. Then,

$$fg \in \mathcal{L}(rQ + (s+s')P) \setminus \mathcal{L}(rQ + (s+s'-1)P).$$

This implies

$$H_0 + H_r \subseteq H_r.$$

Define the set

$$H_r^* = \{s \in \mathbb{Z} | C_{r,s} \neq C_{r,s-1}\}.$$

So we can restrict to consider the codes $C_{r,s}$ with $s \in H_r^*$. It is easy to see that H_r^* consists of n_2 elements. Let us write $H_r^* = \{s_1^* < s_2^* < \dots < s_{n_2}^*\}$. Then $\dim(C_{r,s_i^*}) = i$. Clearly, $H_r^* \subseteq H_r$ and $H_r^* \cap \{s | rq + s < n_2\} = H_r \cap \{s | rq + s < n_2\}$. For $s \in \mathbb{Z}$ satisfying $rq + s \geq n_2$, then $s \in H_r^*$ if and only if $s \in H_r^\perp$, where H_r^\perp is defined by

$$H_r^\perp := \begin{cases} q^5 + q^4 - q^3 - q^2 - 2q + 1 - H_{q^2-1-r} & \text{for } 0 \leq r \leq q^2 - 1, \\ q^5 + q^4 - 2q^3 - 2q^2 - 3q + 1 - H_{2q^2+q-r} & \text{for } q^2 \leq r \leq q^2 + q. \end{cases} \quad (9)$$

We remark that besides the Goppa bound there are several bounds available to estimate the minimum distance of a code. One of the most interesting is the order bound. We can follow the version of [21], which is briefly explained below. For $i = 1, \dots, n_2$, let

$$\Lambda_i^r := \{(a, b) | a \in H_0, b \in H_r, a + b = s_i^* \in H_r^*\}.$$

By using the notation in [21], we consider the infinite sequence $S = P, P, P, \dots$, then the minimum distance of the dual code of $C_{r,s}$ verifies

$$d(C_{r,s}^\perp) \geq d_S(rQ + sP) := \min_{s_i^* > s} \{\#\Lambda_i^r\}.$$

Example 19. Firstly, we consider the code $C_{5,s}$ over \mathbb{F}_8 of length $n_2 = 28$. The genus of curve \mathcal{X} is $g = 6$. We find that the dual code of $C_{5,s}$ is equivalent to $C_{5,18-s}$ by Proposition 15. According to Proposition 3 and Equation (9), we get

$$H_5 = \{-6, -5, -2, -1, 0, 1, 2, 3, \dots\},$$

and

$$H_5^* = \{-6, -5, -2, -1, 0, 1, 2, 3, \dots, 16, 17, 18, 19, 20, 21, 24, 25\}.$$

By direct computation, we obtain the following table.

TABLE II

s	dim	basis	d	s	dim	basis	d
-6	1	x^2y	28	10	15	x^2y^{-3}	8
-5	2	xy	24	11	16	xy^{-3}	8
-2	3	x^2	24	12	17	y^{-3}	8
-1	4	x	20	13	18	$x^{-1}y^{-3}$	7
0	5	1	18	14	19	$x^{-2}y^{-3}$	4
1	6	x^3y^{-1}	18	15	20	xy^{-4}	4
2	7	x^2y^{-1}	16	16	21	y^{-4}	4
3	8	xy^{-1}	16	17	22	$x^{-1}y^{-4}$	4
4	9	y^{-1}	15	18	23	$x^{-2}y^{-4}$	3
5	10	x^3y^{-2}	13	19	24	xy^{-5}	3
6	11	x^2y^{-2}	12	20	25	y^{-5}	3
7	12	xy^{-2}	12	21	26	$x^{-1}y^{-5}$	2
8	13	y^{-2}	11	24	27	y^{-6}	2
9	14	$x^{-1}y^{-2}$	10	25	28	$x^{-1}y^{-6}$	1

Comparing Table II with the reference [22], we find the following codes over \mathbb{F}_8 with the best known parameters:

[28, 1, 28], [28, 2, 24], [28, 3, 24], [28, 8, 16], [28, 12, 12], [28, 17, 8], [28, 25, 3], [28, 26, 2], [28, 27, 2], [28, 28, 1].

Table II enables one to construct the codes explicitly. For instance, a [28, 8, 16]-code is constructed by the basis $x^2y, xy, x^2, x, 1, x^3y^{-1}, x^2y^{-1}, xy^{-1}$.

Example 20. Fix $r = 0$, we consider the one-point code $C_{0,s}$ over \mathbb{F}_8 which is dual to $C_{3,32-s}$ up to equivalence. Counting the lattice points in both sets $\Omega_{0,s}$ and $\Omega_{3,s}$, we find the Weierstrass sets

$$H_0 = \{0, 4, 7, 8, 9, 11, 12, 13, 14, \dots\},$$

and

$$H_3 = \{-5, -1, 0, 2, 3, 4, 6, 7, 8, \dots\}.$$

Using Equation (9), the set H_0^* is

$$H_0^* = \{0, 4, 7, 8, 9, 11, 12, 13, 14, \dots, 26, 27, 29, 30, 31, 33, 34, 38\}.$$

TABLE III

s	dim	basis	d	s	dim	basis	d
0	1	1	28	20	15	y^{-5}	8
4	2	y^{-1}	24	21	16	$x^{-1}y^{-5}$	7
7	3	xy^{-2}	21	22	17	$x^{-2}y^{-5}$	7
8	4	y^{-2}	20	23	18	$x^{-3}y^{-5}$	6
9	5	$x^{-1}y^{-2}$	19	24	19	y^{-6}	4
11	6	xy^{-3}	18	25	20	$x^{-1}y^{-6}$	4
12	7	y^{-3}	16	26	21	$x^{-2}y^{-6}$	4
13	8	$x^{-1}y^{-3}$	15	27	22	$x^{-3}y^{-6}$	4
14	9	$x^{-2}y^{-3}$	14	29	23	$x^{-1}y^{-7}$	4
15	10	xy^{-4}	13	30	24	$x^{-2}y^{-7}$	3
16	11	y^{-4}	12	31	25	$x^{-3}y^{-7}$	3
17	12	$x^{-1}y^{-4}$	12	33	26	$x^{-1}y^{-8}$	2
18	13	$x^{-2}y^{-4}$	11	34	27	$x^{-2}y^{-8}$	2
19	14	xy^{-5}	9	38	28	$x^{-2}y^{-9}$	1

We find a $[28, 12, 12]$ -code over \mathbb{F}_8 with the best known parameters as in the previous example.

Example 21. Let us consider the code $C_{4,s}$ over \mathbb{F}_{27} , with $g = 37$, $n_2 = 234$. The code $C_{4,s}$ is dual to $C_{4,282-s}$ up to equivalence. The Weierstrass set for $r = 4$ is $H_4 = \{-10, -1, 0, 8, 9, 16, 17, 18, 19, 25, 26, 27, 28, 29, 34, 35, 36, 37, 38, 39, 42, 43, 44, 45, 46, 47, 48, 51, 52, 53, 54, 55, 56, 57, 58, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, \dots\}$.

According to Equation (9), we obtain $H_4 = \{-10, -1, 0, 8, 9, 16, 17, 18, 19, 25, 26, 27, 28, 29, 34, 35, 36, 37, 38, 39, 42, 43, 44, 45, 46, 47, 48, 51, 52, 53, 54, 55, 56, 57, 58, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, \dots, 214, 215, 216, 217, 218, 219, 220, 221, 222, 223, 225, 226, 227, 228, 229, 230, 231, 232, 235, 236, 237, 238, 239, 240, 241, 244, 245, 246, 247, 248, 249, 254, 255, 256, 257, 258, 264, 265, 266, 267, 274, 275, 283, 284, 293\}$. Computing the order bound, we get $d(C_{4,165}) \geq 59$. So we obtain a new record-giving $[234, 141, \geq 59]$ -code over \mathbb{F}_{27} (according to the tables [23]). Similarly, for general r , we can find the record-giving codes over \mathbb{F}_{27} as follows: $[234, 143, \geq 57]$, $[234, 144, \geq 56]$, $[234, 145, \geq 55]$.

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